Rotating Limepy: an interim report

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Goal: construction of axisymmetric, differentially rotating, anisotropic equilibria within the Limepy framework

Distribution function:

$$f(E, J_z) = A\, E_\gamma \left(g, \frac{E - \Phi(r_t)}{s^2} \right) \exp\left(\frac{-\omega J_z}{s^2} \right) \tag{1}$$

for $E \leq \Phi(r_t)$, vanishing otherwise.

Note-1: the case with $g = 0$ (Woolley-like truncation), corresponds to the family of rotating models proposed by Prendergast & Tomer 1970.

$$f(E, J_z) = A \exp\left(\frac{E - \Phi(r_t)}{s^2} \right) \exp\left(\frac{-\omega J_z}{s^2} \right)$$

the case with $g = 1$ (King-like truncation), corresponds to the family of "rotating King models" (Jarvis & Freedman 1984, Lagoute & Longaretti 1996, Einsel & Spurzem 1999).

$$f(E, J_z) = A \left[\exp\left(\frac{E - \Phi(r_t)}{s^2} \right) - 1 \right] \exp\left(\frac{-\omega J_z}{s^2} \right)$$

Note-2: a further extension to include a dependency on J (as approximate third integral) may be considered (with Lupton & Gunn 1987 as $g = 1$ particular case).

Dimensionless parameters:
1. rotation strength

$$\hat{\omega} = \frac{\omega}{\sqrt{4\pi G \rho_0}} = \frac{\omega r_s}{3s} \tag{2}$$

2. central concentration: $\hat{\phi}_0$
3. order of truncation prescription: g

Physical scales:
A phase space normalization
s energy scale

Dimensionless formulation (general notation consistent with Gieles & Zocchi 2015):
$\hat{E} = \hat{\phi} - \hat{k}$
$\hat{\phi}(\hat{r}) = [\phi(r_t) - \phi(\hat{r})]/s^2$
$\hat{k} = v^2/2s^2$
$\hat{\rho}(\hat{r}) = \rho(\hat{r})/\rho_0$
$\hat{r} = r/r_s$
$\bar{A} = A(2\pi)^{3/2} s^3$

Significant variable substitutions
$v_\varphi = v \cos\mu$ $\qquad d^3v = v^2 \sin\mu \, dv \, d\mu \, d\nu$
$v_\theta = v \sin\mu \cos\nu$
$v_r = v \sin\mu \sin\nu$

$t = \cos\mu$ $\qquad dt = -\sin\mu d\mu$

Calculation of the associated moments in velocity space: dependence on spatial coordinates $(\hat{r}, \theta)$ both implicit (via $\hat{\phi}$) and explicit, via:

$$\frac{\omega J_z}{s^2} = 3\sqrt{2}\,\hat{\omega}\,\hat{r}\,\sin\theta\,t\,\hat{k}^{1/2} = \hat{\omega}\,Q(\hat{r},\theta)\,t\,\hat{k}^{1/2}$$

$$Q(\hat{r},\theta) = 3\sqrt{2}\,\hat{r}\,\sin\theta$$

Density:

$$\rho = \int_{E\leq\phi(r_t)} d^3v f = \frac{\bar{A}}{\sqrt{\pi}}\frac{2}{\hat{\omega}Q}\int_0^{\hat{\phi}} d\hat{k} E_\gamma(g,\hat{\phi}-\hat{k})\sinh(\hat{\omega}Q\hat{k}^{1/2}) \tag{3}$$

Note-3: the non-rotating limit is easily verified since $\sinh(\hat{\omega}Q\hat{k}^{1/2})/\hat{\omega}Q \rightarrow \hat{k}^{1/2}$ as $\hat{\omega}\rightarrow 0$
Note-4: in the case of $g=1$, consistency check with Lagoute & Longaretti 1996:

$$\rho = \frac{\bar{A}}{\sqrt{\pi}}\frac{2}{\hat{\omega}^3Q^3}e^{\hat{\phi}}\int_0^{\hat{\phi}} d\hat{k}\,e^{-\hat{k}}[\hat{\omega}Q\hat{k}^{1/2}\cosh(\hat{\omega}Q\hat{k}^{1/2}) - \sinh(\hat{\omega}Q\hat{k}^{1/2})]$$

First order moment:

$$\begin{aligned}\rho < v_\varphi >= &\int_{E\leq\phi(r_t)} d^3v f v_\varphi \\ &= \frac{\bar{A}}{\sqrt{\pi}}\sqrt{2}s\frac{2}{\hat{\omega}^2Q^2}\int_0^{\hat{\phi}} d\hat{k}\,E_\gamma(g,\hat{\phi}-\hat{k})[\sinh(\hat{\omega}Q\hat{k}^{1/2}) - \hat{\omega}Q\hat{k}^{1/2}\cosh(\hat{\omega}Q\hat{k}^{1/2})]\end{aligned} \tag{4}$$

Note-5: asymptotics for small $\hat{r}$ confirms the solid-body rotation behavior in the central regions

$$\rho < v_\varphi >\sim \frac{\bar{A}}{\sqrt{\pi}}2\sqrt{2}s\int_0^{\hat{\phi}_0} d\hat{k}\,E_\gamma(g,\hat{\phi}_0-\hat{k})\frac{2}{3}\hat{k}^{1/2}Q\hat{\omega} = \bar{A}\sqrt{2}s\frac{2}{3}E_\gamma(g+3/2,\hat{\phi}_0)\hat{\omega}Q \tag{5}$$

Note-6: for symmetry considerations, $< v_\theta >=< v_r >= 0$

Second order moments:

$$\rho < v^2 >= \int_{E\leq\phi(r_t)} d^3v f v^2 = \frac{\bar{A}}{\sqrt{\pi}}\frac{s^2}{\sqrt{2}}\frac{1}{\hat{\omega}Q}\int_0^{\hat{\phi}} d\hat{k}\,\hat{k}E_\gamma(g,\hat{\phi}-\hat{k})\sinh(\hat{\omega}Q\hat{k}^{1/2}) \tag{6}$$

$$\begin{aligned}\rho < v_\varphi^2 >= &\int_{E\leq\phi(r_t)} d^3v f v_\varphi^2 = \\ = \frac{\bar{A}}{\sqrt{\pi}}\sqrt{2}s^2\frac{1}{\hat{\omega}^3Q^3}&\int_0^{\hat{\phi}} d\hat{k}\,E_\gamma(g,\hat{\phi}-\hat{k})[(2+\hat{\omega}^2Q^2\hat{k})\sinh(\hat{\omega}Q\hat{k}^{1/2}) - 2\hat{\omega}Q\hat{k}^{1/2}\cosh(\hat{\omega}Q\hat{k}^{1/2})]\end{aligned} \tag{7}$$

$$\begin{aligned}\rho < v_r^2 >= &\int_{E\leq\phi(r_t)} d^3v f v_r^2 = \\ &= \frac{\bar{A}}{\sqrt{\pi}}\sqrt{2}s^2\frac{1}{\hat{\omega}^3Q^3}\int_0^{\hat{\phi}} d\hat{k}\,E_\gamma(g,\hat{\phi}-\hat{k})[\sinh(\hat{\omega}Q\hat{k}^{1/2}) - \hat{\omega}Q\hat{k}^{1/2}\cosh(\hat{\omega}Q\hat{k}^{1/2})]\end{aligned} \tag{8}$$

Note-7: $\rho < v_\varphi^2 > +2\rho < v_r^2 >= \rho < v^2 >$
Note-8: $\rho < v_\theta^2 >= \rho < v_r^2 >$

Calculation of the potential: self-consistent solution of the (2D) Poisson equation via spectral iteration method

- Expansion in Legendre series of density and potential:

$$\phi^{(n)}(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \phi_l^{(n)}(\hat{r}) U_l(\cos\theta) \qquad \hat{\rho}^{(n)}(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \hat{\rho}_l^{(n)}(\hat{r}) U_l(\cos\theta) \tag{9}$$

$$\hat{\rho}_l = \frac{2l+1}{l} \int_{-1}^{+1} \hat{\rho}(\hat{\mathbf{r}}) U_l(\cos\theta) d(\cos\theta) \tag{10}$$

- Reduced radial problems for the coefficients $\phi_l^{(n)}(\hat{r})$

$$\left[\frac{d^2}{d\hat{r}^2} + \frac{2}{\hat{r}}\frac{d}{d\hat{r}} - \frac{l(l+1)}{\hat{r}^2} \right] \phi_l^{(n)} = -\frac{9}{\hat{\rho}_0} \hat{\rho}_l^{(n-1)} \tag{11}$$

- Boundary Conditions:

$$\phi_0^{(n)}(0) = \phi_0\sqrt{2} \;\; , \;\; \phi_l^{(n)}(0) = 0 \quad \phi_0^{(n)\prime}(0) = \phi_l^{(n)\prime}(0) = 0$$

- Equivalent expression in integral form, via usual Green functions for 2D Poisson equation (by using the method of variation of arbitrary constants)

$$\phi_0^{(n)}(\hat{r}) = \phi_0\sqrt{2} - \frac{9}{\hat{\rho}_0} \left[\int_0^{\hat{r}} \hat{r}' \hat{\rho}_0^{(n-1)}(\hat{r}') d\hat{r}' - \frac{1}{\hat{r}} \int_0^{\hat{r}} \hat{r}'^2 \hat{\rho}_0^{(n-1)}(\hat{r}') d\hat{r}' \right] \tag{12}$$

$$\phi_l^{(n)}(\hat{r}) = \frac{9}{(2l+1)\hat{\rho}_0} \left[\hat{r}^l \int_{\hat{r}}^{\infty} \hat{r}'^{1-l} \hat{\rho}_l^{(n-1)}(\hat{r}') d\hat{r}' + \frac{1}{\hat{r}^{l+1}} \int_0^{\hat{r}} \hat{r}'^{l+2} \hat{\rho}_l^{(n-1)}(\hat{r}') d\hat{r}' \right] \tag{13}$$

- Pseudo-code:

  1. Calculation of the spherical seed solution for the potential (e.g., non-rotating model with same $\hat{\psi}_0$)
  2. Evaluation of the density on the spherical grid: $\hat{\rho}_{ij}$ (e.g., double gaussian quadrature)
  3. Calculation of the density radial coefficients $\hat{\rho}_l(\hat{r})$ (e.g., Discrete Legendre Transform Forward [DLTF])
  4. Calculation of the potential radial coefficients in integral form $\phi_l(\hat{r})$ (e.g., Romberg integration)
  5. Calculation of the potential on the spherical grid ϕ_{ij} (e.g., Discrete Legendre Transform Inverse [DLTI])
  6. Convergence test on the potential: if reached, stop; otherwise repeat from 2.

Still to do:

- Re-check everything, in quiet evening
- Asymptotics of remaining moments for small $\hat{r}$ (i.e., central regions)
- Asymptotics of all moments for small $\hat{\phi}$ (i.e., proximity of the trucation)
- Implementation in Limepy solver
- Validation of numerical solution of Cauchy problems for the coefficients (via stability criteria, and against asymptotics expressions for the full moments)
- Selection of optimal trucation order of the Legendre series (via parameter space exploration, especially for highly rotating models; likely to be $l_{max} < 8$)